GENERALIZED MORREY REGULARITY FOR PARABOLIC EQUATIONS WITH DISCONTINUITY DATA

VAGIF S. GULIYEV[1], LUBOMIRA G. SOFTOVA[2]

Abstract. We obtain continuity in generalized parabolic Morrey spaces of sublinear integrals generated by the parabolic Calderón-Zygmund operator and its commutator with VMO functions. The obtained estimates are used to study global regularity of the solutions of the Cauchy-Dirichlet problem for linear uniformly parabolic equations with discontinuous coefficients.

1. Introduction

The classical Morrey spaces $L_{p,\lambda}$ are originally introduced in [17] in order to prove local Hölder continuity of solutions to certain systems of partial differential equations (PDE's). A real valued function f is said to belong to the Morrey space $L_{p,\lambda}$ with $p \in [1,\infty)$, $\lambda \in (0,n)$ provided the following norm is finite

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \left(\sup_{(x,r)\in\mathbb{R}^n\times\mathbb{R}_+} \frac{1}{r^\lambda} \int_{\mathcal{B}_r(x)} |f(y)|^p \, dy \right)^{1/p}.$$

The main result connected with these spaces is the following celebrated lemma: let $|Df| \in L_{p,\lambda}$ even locally, with $\lambda < p$, then u is Hölder continuous of exponent $\alpha = 1 - \frac{\lambda}{p}$. This result has found many applications in the study the regularity of the strong solutions to elliptic and parabolic PDE's and systems. In [5] Chiarenza and Frasca showed boundedness of the Hardy-Littlewood maximal operator in $L_{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity in that spaces of some classical integral operators. These operators appear in the representation formulas of the solutions of linear PDE's and systems. Thus the results in [5] permit to study the regularity of the solutions of these operators in $L_{p,\lambda}$ (see [20, 23]). In [16] Mizuhara extends the concept of Morrey of integral average over a ball with a certain growth, taking a weight function $\omega(x,r) : \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ. Thus he put the beginning of the study of the generalized Morrey spaces $L_{p,\omega}$ under various conditions on the weight function. In [18] Nakai extended the results of [5] in $L_{p,\omega}$ imposing the following conditions on the weight

$$\int_r^\infty \frac{\omega(x,s)}{s^{n+1}} ds \le C \frac{\omega(x,r)}{r^n}, \qquad C_1 \le \frac{\omega(x,s)}{\omega(x,r)} \le C_2 \quad r \le s \le 2r,$$

where the constants do not depend on s, r and x. In [22, 24, 25] global $L_{p,\omega}$-regularity of solutions to elliptic and parabolic boundary value problems is obtained using explicit representation formula.

1991 *Mathematics Subject Classification.* 35K15; 42B20.

Key words and phrases. Generalized parabolic Morrey spaces; sublinear integrals; parabolic Calderón - Zygmund integrals; commutators; BMO; VMO; parabolic equations; Cauchy-Dirichlet problem.

Other generalizations of the Morrey spaces are considered in [2, 8, 9, 11] where the continuity of sublinear operators generated by various classical integral operators as the Calderón-Zygmund, Riesz and others is proved. In [12] we have applied these results to the study of regularity of solutions to the Dirichlet problem for linear uniformly elliptic equations.

In the present work we obtain global regularity of the solutions of the Cauchy-Dirichlet problem for parabolic non-divergence equations with VMO coefficients in $M_{p,\varphi}$. This problem has been studied in the framework of the Morrey spaces in [19] and in the weighted Lebesgue spaces in [10]. Here we extend these results in $M_{p,\varphi}$. For this goal we study continuity in $M_{p,\varphi}$ of sublinear operators generated by the Calderón-Zygmund integrals with parabolic kernels and their commutators with BMO functions (Section 3). The last ones enter in the interior representation formula of the derivatives $D_{ij}u$ of the solution of (2.1). In Section 4 we establish continuity for sublinear integrals generated by nonsingular integral operators and commutators. These integrals enter in the boundary representation formula for $D_{ij}u$. The global a priori estimate for u is obtained in Section 6.

Throughout this paper the following notations will be used:

- $x=(x',t), y=(y',\tau)\in\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R},\ \mathbb{R}^{n+1}_+=\mathbb{R}^n\times\mathbb{R}_+$;
- $x=(x'',x_n,t)\in\mathbb{D}^{n+1}_+=\mathbb{R}^{n-1}\times\mathbb{R}_+\times\mathbb{R}_+,\ \mathbb{D}^{n+1}_-=\mathbb{R}^{n-1}\times\mathbb{R}_-\times\mathbb{R}_+$;
- $|\cdot|$ is the Euclidean metric, $|x|=\left(\sum_{i=1}^n x_i^2+t^2\right)^{1/2}$;
- $D_iu=\partial u/\partial x_i,\ Du=(D_1u,\dots,D_nu),\ u_t=\partial u/\partial t$;
- $D_{ij}u=\partial^2u/\partial x_i\partial x_j,\ D^2u=\{D_{ij}u\}_{ij=1}^n$ means the Hessian matrix of u;
- $\mathcal{B}_r(x')=\{y'\in\mathbb{R}^n:\ |x'-y'|<r\}$, $|\mathcal{B}_r|=Cr^n$;
- $\mathcal{I}_r(x)=\{y\in\mathbb{R}^{n+1}:\ |x'-y'|<r, |t-\tau|<r^2\}$, $|\mathcal{I}_r|=Cr^{n+2}$;
- $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$;
- for any $f\in L_p(A)$, $A\subset\mathbb{R}^{n+1}$ we write
$$\|f\|_{p,A}\equiv\|f\|_{L_p(A)}=\left(\int_A|f(y)|^pdy\right)^{1/p}.$$
- The standard summation convention on repeated upper and lower indexes is adopted.
- The letter C is used for various positive constants and may change from one occurrence to another.

2. Definitions and statement of the problem

In the following, besides the standard parabolic metric $\varrho(x)=\max(|x'|,|t|^{1/2})$ we use the equivalent one $\rho(x)=\left(\frac{|x'|^2+\sqrt{|x'|^4+4t^2}}{2}\right)^{1/2}$ introduced by Fabes and Riviére in [7]. The induced by it topology consists of ellipsoids
$$\mathcal{E}_r(x)=\left\{y\in\mathbb{R}^{n+1}:\ \frac{|x'-y'|^2}{r^2}+\frac{|t-\tau|^2}{r^4}<1\right\},\ |\mathcal{E}_r|=Cr^{n+2},\ \mathcal{E}_1(x)\equiv\mathcal{B}_1(x)\,.$$
It is easy to see that the metrics $\rho(\cdot)$ and $\varrho(\cdot)$ are equivalent. Infact for each $\mathcal{E}_r$ there exist parabolic cylinders $\underline{\mathcal{I}}$ and $\overline{\mathcal{I}}$ with measure comparable to r^{n+2} such that $\underline{\mathcal{I}}\subset\mathcal{E}_r\subset\overline{\mathcal{I}}$. In what follows all estimate obtained over ellipsoids hold true also over parabolic cylinders and we shall use this property without explicit references.

Let $\Omega\subset\mathbb{R}^n$ be a bounded $C^{1,1}$-domain and $Q=\Omega\times(0,T)$, $T>0$ be a cylinder in $\mathbb{R}^{n+1}_+$. We give the definitions of the functional spaces which we are going to use.

Definition 2.1. *Let $a \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $a_{\mathcal{E}_r} = |\mathcal{E}_r|^{-1} \int_{\mathcal{E}_r} a(y)dy$ be the mean integral of a. Denote*

$$\eta_a(R) = \sup_{r \le R} \frac{1}{|\mathcal{E}_r|} \int_{\mathcal{E}_r} |f(y) - f_{\mathcal{E}_r}| dy \quad \text{for every } R > 0$$

where $\mathcal{E}_r$ ranges over all ellipsoids in $\mathbb{R}^{n+1}$. We say that

- *$a \in BMO$ (bounded mean oscillation, [13]) provided the following is finite*

$$\|a\|_* = \sup_{R>0} \eta_a(R).$$

The quantity $\|\cdot\|_*$ is a norm in BMO modulo constant function under which BMO is a Banach space.

- *$a \in VMO$ (vanishing mean oscillation, [21]) if $a \in BMO$ and*

$$\lim_{R\to 0} \eta_a(R) = 0.$$

The quantity $\eta_a(R)$ is called VMO-modulus of a.

For any bounded cylinder Q we define $BMO(Q)$ and $VMO(Q)$ taking $a \in L_1(Q)$ and Q_r instead of $\mathcal{E}_r$ in the definition above.

According to [1, 14], having a function $a \in BMO(Q)$ or $VMO(Q)$ it is possible to extend it in the whole $\mathbb{R}^{n+1}$ preserving its BMO-norm or VMO-modulus, respectively. In the following we use this property without explicit references. Any bounded uniformly continuous (BUC) function f with modulus of continuity $\omega_f(R)$ belongs to VMO with $\eta_f(R) = \omega_f(R)$. Besides that, BMO and VMO contain also discontinuous functions and the following example shows the inclusion $W_{1,n+2}(\mathbb{R}^{n+1}) \subset VMO \subset BMO$.

Example 2.2. $f_\alpha(x) = |\log \rho(x)|^\alpha \in VMO$ *for any* $\alpha \in (0,1)$;
$f_\alpha \in W_{1,n+2}(\mathbb{R}^{n+1})$ *for* $\alpha \in (0, 1 - 1/(n+2))$;
$f_\alpha \notin W_{1,n+2}(\mathbb{R}^{n+1})$ *for* $\alpha \in [1 - 1/(n+2), 1)$;
$f(x) = |\log \rho(x)| \in BMO \setminus VMO$;
$\sin f_\alpha(x) \in VMO \cap L_\infty(\mathbb{R}^{n+1})$.

Definition 2.3. *Let $\varphi : \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function and $p \in [1, \infty)$. The generalized parabolic Morrey space $M_{p,\varphi}(\mathbb{R}^{n+1})$ consists of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^{n+1})$ such that*

$$\|f\|_{p,\varphi;\mathbb{R}^{n+1}} = \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r)^{-1} \left(r^{-(n+2)} \int_{\mathcal{E}_r(x)} |f(y)|^p dy \right)^{1/p} < \infty.$$

The space $M_{p,\varphi}(Q)$ consists of $L_p(Q)$ functions provided the following norm is finite

$$\|f\|_{p,\varphi;Q} = \sup_{(x,r)\in Q\times\mathbb{R}_+} \varphi(x,r)^{-1} \left(r^{-(n+2)} \int_{Q_r(x)} |f(y)|^p dy \right)^{1/p}$$

where $Q_r(x) = Q \cap \mathcal{I}_r(x)$. The generalized weak parabolic Morrey space $WM_{1,\varphi}(\mathbb{R}^{n+1})$ consists of all measurable functions such that

$$\|f\|_{WM_{1,\varphi}(\mathbb{R}^{n+1})} = \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r)^{-1} r^{-n-2} \|f\|_{WL_1(\mathcal{E}_r(x))}$$

where WL_1 denotes the weak L_1 space.

The generalized Sobolev-Morrey space $W^{2,1}_{p,\varphi}(Q)$, $p\in[1,\infty)$ consist of all Sobolev functions $u\in W^{2,1}_p(Q)$ with distributional derivatives $D^l_tD^s_xu\in M_{p,\varphi}(Q)$, $0\le 2l+|s|\le 2$ endowed by the norm

$$\|u\|_{W^{2,1}_{p,\varphi}(Q)}=\|u_t\|_{p,\varphi;Q}+\sum_{|s|\le 2}\|D^su\|_{p,\varphi;Q}.$$

$$\overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q)=\big\{u\in W^{2,1}_{p,\varphi}(Q):\ u(x)=0, x\in\partial Q\big\},\ \|u\|_{\overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q)}=\|u\|_{W^{2,1}_{p,\varphi}(Q)}$$

where ∂Q means the parabolic boundary $\Omega\cup(\partial\Omega\times(0,T))$.

We consider the Cauchy-Dirichlet problem for linear parabolic equation

$$\begin{cases} u_t-a^{ij}(x)D_{ij}u(x)=f(x) \text{ a.a. } x\in Q, \quad u\in\overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q)\end{cases} \tag{2.1}$$

where the coefficient matrix $\mathbf{a}(x)=\{a^{ij}(x)\}^n_{i,j=1}$ satisfies

$$\begin{cases}\exists\ \Lambda>0:\ \Lambda^{-1}|\xi|^2\le a^{ij}(x)\xi_i\xi_j\le\Lambda|\xi|^2 \text{ for a.a. } x\in Q,\ \forall\xi\in\mathbb{R}^n\\ a^{ij}(x)=a^{ji}(x) \text{ that implies } a^{ij}\in L_\infty(Q).\end{cases} \tag{2.2}$$

Theorem 2.4. (Main result) *Let $\mathbf{a}\in VMO(Q)$ satisfy (2.2) and for each $p\in(1,\infty)$, $u\in\overset{\circ}{W}{}^{2,1}_p(Q)$ be a strong solution of (2.1). If $f\in M_{p,\varphi}(Q)$ with $\varphi(x,r)$ being measurable positive function satisfying*

$$\int_r^\infty\Big(1+\ln\frac{s}{r}\Big)\frac{\operatorname*{essinf}_{s<\zeta<\infty}\varphi(x,\zeta)\zeta^{\frac{n+2}{p}}}{s^{\frac{n+2}{p}+1}}\,ds\le C\,\varphi(x,r),\quad (x,r)\in Q\times\mathbb{R}_+ \tag{2.3}$$

then $u\in\overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q)$ and

$$\|u\|_{\overset{\circ}{W}{}^{2,1}_{p,\varphi}(Q)}\le C\|f\|_{p,\varphi;Q} \tag{2.4}$$

with $C=C(n,p,\Lambda,\partial\Omega,T,\|\mathbf{a}\|_{\infty;Q},\eta_a)$.

3. Sublinear operators generated by parabolic singular integrals in generalized Morrey spaces

Let $f\in L_1(\mathbb{R}^{n+1})$ be a function with a compact support and $a\in BMO$. For any $x\notin\operatorname{supp}f$ define the sublinear operators T and T_a such that

$$|Tf(x)|\le C\int_{\mathbb{R}^{n+1}}\frac{|f(y)|}{\rho(x-y)^{n+2}}\,dy \tag{3.5}$$

$$|T_af(x)|\le C\int_{\mathbb{R}^{n+1}}|a(x)-a(y)|\,\frac{|f(y)|}{\rho(x-y)^{n+2}}\,dy. \tag{3.6}$$

Suppose in addition that the both operators are bounded in $L_p(\mathbb{R}^{n+1})$ satisfying the estimates

$$\|Tf\|_{p;\mathbb{R}^{n+1}}\le C\|f\|_{p;\mathbb{R}^{n+1}},\quad \|T_af\|_{p;\mathbb{R}^{n+1}}\le C\|a\|_*\|f\|_{p;\mathbb{R}^{n+1}} \tag{3.7}$$

with constants independent of a and f. The following known result concerns the Hardy operator $Hg(r)=\frac{1}{r}\int_0^r g(s)ds$, $r>0$.

Theorem 3.1. ([4]) *The inequality*

$$\operatorname*{esssup}_{r>0} w(r)Hg(r) \le A \operatorname*{esssup}_{r>0} v(r)g(r) \tag{3.8}$$

holds for all non-increasing functions $g:\mathbb{R}_+\to\mathbb{R}_+$ if and only if

$$A = C\sup_{r>0}\frac{w(r)}{r}\int_0^r \frac{ds}{\operatorname*{esssup}_{0<\zeta<s} v(\zeta)} < \infty. \tag{3.9}$$

Lemma 3.2. *Let $f\in L_p^{\rm loc}(\mathbb{R}^{n+1})$, $p\in[1,\infty)$ be such that*

$$\int_r^\infty s^{-\frac{n+2}{p}-1}\|f\|_{p;\mathcal{E}_s(x_0)}ds<\infty \quad \forall\ (x_0,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+ \tag{3.10}$$

and T be a sublinear operator satisfying (3.5).

(i) *If $p>1$ and T bounded on $L_p(\mathbb{R}^{n+1})$ then*

$$\|Tf\|_{p;\mathcal{E}_r(x_0)} \le C\, r^{\frac{n+2}{p}} \int_{2r}^\infty s^{-\frac{n+2}{p}-1}\|f\|_{p;\mathcal{E}_s(x_0)}ds. \tag{3.11}$$

(ii) *If $p=1$ and T bounded from $L_1(\mathbb{R}^{n+1})$ on $WL_1(\mathbb{R}^{n+1})$ then*

$$\|Tf\|_{WL_1(\mathcal{E}_r(x_0))} \le Cr^{n+2}\int_{2r}^\infty s^{-n-3}\|f\|_{1,\mathcal{E}_s(x_0)}\,ds \tag{3.12}$$

where the constants are independent of r, x_0 and f.

Proof. (i) Fix a point $x_0\in\mathbb{R}^{n+1}$ and consider an ellipsoid $\mathcal{E}_r(x_0)$. Denote by $2\mathcal{E}_r(x_0)=\mathcal{E}_{2r}(x_0)$ and $\mathcal{E}_r^c(x_0)=\mathbb{R}^{n+1}\setminus\mathcal{E}_r(x_0)$. Consider the decomposition of f with respect to the ellipsoid $\mathcal{E}_r(x_0)$

$$f=f\chi_{2\mathcal{E}_r(x_0)}+f\chi_{2\mathcal{E}_r^c(x_0)}=f_1+f_2.$$

Because of the (p,p)-boundedness of the operator T and $f_1\in L_p(\mathbb{R}^{n+1})$ we have

$$\|Tf_1\|_{p;\mathcal{E}_r(x_0)}\le\|Tf_1\|_{p;\mathbb{R}^{n+1}}\le C\|f_1\|_{p;\mathbb{R}^{n+1}}=C\|f\|_{p;2\mathcal{E}_r(x_0)}.$$

It is easy to see that for arbitrary points $x\in\mathcal{E}_r(x_0)$ and $y\in2\mathcal{E}_r^c(x_0)$ it holds

$$\frac12\rho(x_0-y)\le\rho(x-y)\le\frac32\rho(x_0-y). \tag{3.13}$$

Applying (3.5), (3.13), the Fubini theorem and the Hölder inequality to Tf_2 we get

$$\begin{aligned}|Tf_2(x)|\le& C\int_{2\mathcal{E}_r^c(x_0)}\frac{|f(y)|}{\rho(x_0-y)^{n+2}}dy\le C\int_{2\mathcal{E}_r^c(x_0)}|f(y)|\left(\int_{\rho(x_0-y)}^\infty\frac{ds}{s^{n+3}}\right)dy\\ \le& C\int_{2r}^\infty\left(\int_{2r\le\rho(x_0-y)<s}|f(y)|dy\right)\frac{ds}{s^{n+3}}\\ \le& C\int_{2r}^\infty\left(\int_{\mathcal{E}_s(x_0)}|f(y)|dy\right)\frac{ds}{s^{n+3}}\le C\int_{2r}^\infty\|f\|_{p;\mathcal{E}_s(x_0)}\frac{ds}{s^{\frac{n+2}{p}+1}}.\end{aligned}$$

Direct calculations give

$$\|Tf_2\|_{p,\mathcal{E}_r(x_0)}\le Cr^{\frac{n+2}{p}}\int_{2r}^\infty\|f\|_{p;\mathcal{E}_s(x_0)}\frac{ds}{s^{\frac{n+2}{p}+1}} \tag{3.14}$$

which holds for all $p \in [1,\infty)$. Thus

$$\|Tf\|_{p;\mathcal{E}_r(x_0)} \leq C \left(\|f\|_{p;2\mathcal{E}_r(x_0)} + r^{\frac{n+2}{p}} \int_{2r}^{\infty} \|f\|_{p;\mathcal{E}_s(x_0)} \frac{ds}{s^{\frac{n+2}{p}+1}} \right). \tag{3.15}$$

On the other hand

$$\|f\|_{p,2\mathcal{E}_r(x_0)} \leq C r^{\frac{n+2}{p}} \int_{2r}^{\infty} \|f\|_{p;\mathcal{E}_s(x_0)} \frac{ds}{s^{\frac{n+2}{p}+1}} \tag{3.16}$$

which unified with (3.15) gives (3.11).

(ii) Let $f \in L_1(\mathbb{R}^{n+1})$, the weak $(1,1)$-boundedness of T implies

$$\begin{aligned} \|Tf_1\|_{WL_1(\mathcal{E}_r(x_0))} &\leq \|Tf_1\|_{WL_1(\mathbb{R}^{n+1})} \\ &\leq C\|f_1\|_{1,\mathbb{R}^{n+1}} = C\|f\|_{1,2\mathcal{E}_r(x_0)} \\ &\leq Cr^{n+2} \int_{2r}^{+\infty} \|f\|_{1,\mathcal{E}_s(x_0)} \frac{ds}{s^{n+3}} \end{aligned}$$

that unified with (3.14) gives (3.12). □

Theorem 3.3. *Let $p \in [1,\infty)$, $\varphi(x,r)$ be a measurable positive function satisfying*

$$\int_r^{\infty} \frac{\operatorname*{essinf}_{s<\zeta<\infty} \varphi(x,\zeta)\zeta^{\frac{n+2}{p}}}{s^{\frac{n+2}{p}+1}}\, ds \leq C\, \varphi(x,r) \quad \forall\ (x,r) \in \mathbb{R}^{n+1} \times \mathbb{R}_+ \tag{3.17}$$

and T be sublinear operator satisfying (3.5)*.*

(i) *If $p>1$ and T bounded on $L_p(\mathbb{R}^{n+1})$ than T is bounded on $M_{p,\varphi}(\mathbb{R}^{n+1})$ and*

$$\|Tf\|_{p,\varphi;\mathbb{R}^{n+1}} \leq C\|f\|_{p,\varphi;\mathbb{R}^{n+1}}\,. \tag{3.18}$$

(ii) *If $p=1$ and T bounded from $L_1(\mathbb{R}^{n+1})$ to $WL_1(\mathbb{R}^{n+1})$ than it is bounded from $M_{1,\varphi}(\mathbb{R}^{n+1})$ to $WM_{1,\varphi}(\mathbb{R}^{n+1})$ and*

$$\|Tf\|_{WM_{1,\varphi}(\mathbb{R}^{n+1})} \leq C\|f\|_{1,\varphi;\mathbb{R}^{n+1}} \tag{3.19}$$

with constants independent on f.

Proof. (i) By Lemma 3.2 we have

$$\begin{aligned} \|Tf\|_{p,\varphi;\mathbb{R}^{n+1}} &\leq C \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r)^{-1} \int_r^{\infty} \|f\|_{p;\mathcal{E}_s(x)} \frac{ds}{s^{\frac{n+2}{p}+1}} \\ &= C \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r)^{-1} \int_0^{r^{-(n+2)/p}} \|f\|_{p;\mathcal{E}_{s^{-p/(n+2)}}(x)}\, ds \\ &= C \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r^{-p/(n+2)})^{-1} \int_0^{r} \|f\|_{p;\mathcal{E}_{s^{-p/(n+2)}}(x)}\, ds. \end{aligned}$$

Applying the Theorem 3.1 with

$$\begin{aligned} &w(r) = v(r) = r\varphi(x,r^{-p/(n+2)})^{-1}, \quad g(r) = \|f\|_{p;\mathcal{E}_{r^{-p/(n+2)}}(x)}, \\ &Hg(r) = r^{-1}\int_0^r \|f\|_{p;\mathcal{E}_{s^{-p/(n+2)}}(x)}\, ds, \end{aligned}$$

where the condition (3.9) is equivalent to (3.17), we get (3.18).

(ii) Making use of (3.12) and (3.8) we get

$$
\begin{aligned}
\|Tf\|_{WM_{1,\varphi}(\mathbb{R}^{n+1})} &\le C \sup_{(x_0,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x_0,r)^{-1}\int_r^\infty \|f\|_{1,\mathcal{E}_s(x_0)}\frac{ds}{s^{n+3}} \\
&= C \sup_{(x_0,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x_0,r^{-\frac{1}{n+2}})^{-1}\int_0^r \|f\|_{1,\mathcal{E}_{s^{-1/(n+2)}}(x_0)}\,ds \\
&\le C \sup_{(x_0,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x_0,r^{-\frac{1}{n+2}})^{-1} r\|f\|_{1,\mathcal{E}_{r^{-1/(n+2)}}(x_0)} = C\|f\|_{1,\varphi;\mathbb{R}^{n+1}}\,.
\end{aligned}
$$

□

Our next step is to show boundedness of T_a in $M_{p,\varphi}(\mathbb{R}^{n+1})$. For this goal we recall some properties of the BMO functions.

Lemma 3.4. (John-Nirenberg lemma, [3, Lemma 2.8]) *Let $a\in BMO$ and $p\in[1,\infty)$. Then for any $\mathcal{E}_r$ there holds*

$$
\left(\frac{1}{|\mathcal{E}_r|}\int_{\mathcal{E}_r}|a(y)-a_{\mathcal{E}_r}|^p dy\right)^{\frac{1}{p}} \le C(p)\|a\|_*.
$$

As an immediate consequence of Lemma 3.4 we get the following property.

Corollary 3.5. *Let $a\in BMO$ then for all $0<2r<s$ it holds*

$$
|a_{\mathcal{E}_r}-a_{\mathcal{E}_s}| \le C(n)\big(1+\ln\frac{s}{r}\big)\|a\|_*\,. \tag{3.20}
$$

Proof. Since $s>2r$ there exists $k\in\mathbb{N}$, $k\ge1$ such that $2^k r<s\le 2^{k+1}r$ and hence $k\ln 2<\ln\frac{s}{r}\le(k+1)\ln 2$. By [3, Lemma 2.9] we have

$$
\begin{aligned}
|a_{\mathcal{E}_s}-a_{\mathcal{E}_r}| &\le |a_{2^k\mathcal{E}_r}-a_{\mathcal{E}_r}| + |a_{2^k\mathcal{E}_r}-a_{\mathcal{E}_s}| \\
&\le C(n)k\|a\|_* + \frac{1}{|2^k\mathcal{E}_r|}\int_{2^k\mathcal{E}_r}|a(y)-a_{\mathcal{E}_s}|dy \\
&\le C(n)\left(k\|a\|_* + \frac{1}{|\mathcal{E}_s|}\int_{\mathcal{E}_s}|a(y)-a_{\mathcal{E}_s}|dy\right) \\
&< C(n)\big(\ln\frac{s}{r}+1\big)\|a\|_*\,.
\end{aligned}
$$

□

To estimate the norm of T_a we shall employ the same idea which we used in the proof of Lemma 3.2.

Lemma 3.6. *Let $a\in BMO$ and T_a be a bounded operator in $L_p(\mathbb{R}^{n+1})$, $p\in(1,\infty)$ satisfying (3.6) and (3.7). Suppose that for any $f\in L_p^{\mathrm{loc}}(\mathbb{R}^{n+1})$*

$$
\int_r^\infty\Big(1+\ln\frac{s}{r}\Big)\|f\|_{p;\mathcal{E}_s(x_0)}\frac{ds}{s^{\frac{n+2}{p}+1}}<\infty \quad \forall\ (x_0,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+\,. \tag{3.21}
$$

Then

$$
\|T_af\|_{p;\mathcal{E}_r(x_0)} \le C\|a\|_*\, r^{\frac{n+2}{p}}\int_{2r}^\infty\Big(1+\ln\frac{s}{r}\Big)\|f\|_{p;\mathcal{E}_s(x_0)}\frac{ds}{s^{\frac{n+2}{p}+1}} \tag{3.22}
$$

where C is independent of a, f, x_0 and r.

Proof. Fix a point $x_0 \in \mathbb{R}^{n+1}$ and consider the decomposition $f = f\chi_{2\mathcal{E}_r(x_0)} + f\chi_{2\mathcal{E}^c_r(x_0)} = f_1 + f_2$. Hence

$$\|T_a f\|_{p;\mathcal{E}_r(x_0)} \le \|T_a f_1\|_{p;\mathcal{E}_r(x_0)} + \|T_a f_2\|_{p;\mathcal{E}_r(x_0)}$$

and by (3.7) as in Lemma 3.2 we have

$$\|T_a f_1\|_{p;\mathcal{E}_r(x_0)} \le C\|a\|_* \, \|f\|_{p;2\mathcal{E}_r(x_0)}. \tag{3.23}$$

On the other hand, because of (3.13) we can write

$$\begin{aligned}
\|T_a f_2\|_{p;\mathcal{E}_r(x_0)} &\le C\left(\int_{\mathcal{E}_r(x_0)}\left(\int_{2\mathcal{E}^c_r(x_0)} \frac{|a(x)-a(y)||f(y)|}{\rho(x_0-y)^{n+2}}dy\right)^p dx\right)^{\frac{1}{p}}\\
&\le C\left(\int_{\mathcal{E}_r(x_0)}\left(\int_{2\mathcal{E}^c_r(x_0)} \frac{|a(y)-a_{\mathcal{E}_r(x_0)}||f(y)|}{\rho(x_0-y)^{n+2}}dy\right)^p dx\right)^{\frac{1}{p}}\\
&\quad + C\left(\int_{\mathcal{E}_r(x_0)}\left(\int_{2\mathcal{E}^c_r(x_0)} \frac{|a(x)-a_{\mathcal{E}_r(x_0)}||f(y)|}{\rho(x_0-y)^{n+2}}dy\right)^p dx\right)^{\frac{1}{p}}\\
&= I_1 + I_2.
\end{aligned}$$

Applying (3.6), the Fubini theorem and the Hölder inequality as in Lemmate 3.2 and 3.4 we get

$$\begin{aligned}
I_1 &\le Cr^{\frac{n+2}{p}}\left(\int_{2r}^\infty \int_{\mathcal{E}_s(x_0)} |a(y)-a_{\mathcal{E}_r(x_0)}||f(y)|dy\right)\frac{ds}{s^{n+3}}\\
&\le Cr^{\frac{n+2}{p}}\left(\int_{2r}^\infty \int_{\mathcal{E}_s(x_0)} |a(y)-a_{\mathcal{E}_s(x_0)}||f(y)|dy\right)\frac{ds}{s^{n+3}}\\
&\quad + Cr^{\frac{n+2}{p}}\int_{2r}^\infty |a_{\mathcal{E}_r(x_0)}-a_{\mathcal{E}_s(x_0)}|\left(\int_{\mathcal{E}_s(x_0)}|f(y)|dy\right)\frac{ds}{s^{n+3}}\\
&\le Cr^{\frac{n+2}{p}}\int_{2r}^\infty\left(\int_{\mathcal{E}_s(x_0)}|a(y)-a_{\mathcal{E}_s(x_0)}|^{\frac{p}{p-1}}dy\right)^{\frac{p-1}{p}}\|f\|_{p;\mathcal{E}_s(x_0)}\frac{ds}{s^{n+3}}\\
&\quad + Cr^{\frac{n+2}{p}}\int_{2r}^\infty |a_{\mathcal{E}_r(x_0)}-a_{\mathcal{E}_s(x_0)}|\|f\|_{p;\mathcal{E}_s(x_0)}\frac{ds}{s^{\frac{n+2}{p}+1}}\\
&\le C\|a\|_*\, r^{\frac{n+2}{p}}\int_{2r}^\infty\left(1+\ln\frac{s}{r}\right)\|f\|_{p;\mathcal{E}_s(x_0)}\frac{ds}{s^{\frac{n+2}{p}+1}}.
\end{aligned}$$

In order to estimate I_2 we note that

$$I_2 = \left(\int_{\mathcal{E}_r(x_0)}|a(x)-a_{\mathcal{E}_r(x_0)}|^p dx\right)^{\frac{1}{p}}\int_{2\mathcal{E}^c_r(x_0)}\frac{|f(y)|}{\rho(x_0-y)^{n+2}}dy.$$

By Lemma 3.4 and (3.14) we get

$$I_2 \le C\|a\|_*\, r^{\frac{n+2}{p}}\int_{2\mathcal{E}^c_r(x_0)}\frac{|f(y)|}{\rho(x_0-y)^{n+2}}dy \le C\|a\|_*\, r^{\frac{n+2}{p}}\int_{2r}^\infty \|f\|_{p;\mathcal{E}_s(x_0)}\frac{ds}{s^{\frac{n+2}{p}+1}}.$$

Summing up (3.23), I_1 and I_2 we get

$$\|T_a f\|_{p;\mathcal{E}_r(x_0)} \le C\|a\|_*\left(\|f\|_{p;2\mathcal{E}_r(x_0)} + r^{\frac{n+2}{p}}\int_{2r}^\infty\left(1+\ln\frac{s}{r}\right)\|f\|_{p;\mathcal{E}_s(x_0)}\frac{ds}{s^{\frac{n+2}{p}+1}}\right)$$

and the statement follows after applying (3.16). □

Theorem 3.7. *Let $p \in (1,\infty)$ and $\varphi(x,r)$ be measurable positive function such that*

$$\int_r^\infty \Big(1+\ln\frac{s}{r}\Big)\frac{\operatorname*{essinf}_{s<\zeta<\infty}\varphi(x,\zeta)\zeta^{\frac{n+2}{p}}}{s^{\frac{n+2}{p}+1}}\,ds \le C\,\varphi(x,r), \quad \forall\ (x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+ \tag{3.24}$$

where C is independent of x and r. Suppose $a \in BMO$ and T_a be sublinear operator satisfying (3.6). If T_a is bounded in $L_p(\mathbb{R}^{n+1})$, then it is bounded in $M_{p,\varphi}(\mathbb{R}^{n+1})$ and

$$\|T_a f\|_{p,\varphi;\mathbb{R}^{n+1}} \le C\|a\|_*\,\|f\|_{p,\varphi;\mathbb{R}^{n+1}} \tag{3.25}$$

with a constant independent of a and f.

The statement of the theorem follows by Lemma 3.6 and Theorem 3.1 in the same manner as the Theorem 3.3.

Example 3.8. *The functions $\varphi(x,r)=r^{\beta-\frac{n+2}{p}}$ and $\varphi(x,r)=r^{\beta-\frac{n+2}{p}}\log^m(e+r)$ with $0<\beta<\frac{n+2}{p}$ and $m\ge 1$ are weight functions satisfying the condition (3.24).*

4. Sublinear operators generated by nonsingular integrals in generalized Morrey spaces

For any $x\in\mathbb{D}_+^{n+1}$ define $\widetilde{x}=(x'',-x_n,t)\in\mathbb{D}_-^{n+1}$ and $x^0=(x'',0,0)\in\mathbb{R}^{n-1}$. Consider the semi-ellipsoids $\mathcal{E}_r^+(x^0)=\mathcal{E}_r(x^0)\cap\mathbb{D}_+^{n+1}$. Let $f\in L_1(\mathbb{D}_+^{n+1})$, $a\in BMO(\mathbb{D}_+^{n+1})$ and $\widetilde{T}$ and $\widetilde{T}_a$ be sublinear operators such that

$$|\widetilde{T}f(x)| \le C\int_{\mathbb{D}_+^{n+1}}\frac{|f(y)|}{\rho(\widetilde{x}-y)^{n+2}}\,dy \tag{4.26}$$

$$|\widetilde{T}_a f(x)| \le C\int_{\mathbb{D}_+^{n+1}}|a(x)-a(y)|\,\frac{|f(y)|}{\rho(\widetilde{x}-y)^{n+2}}dy. \tag{4.27}$$

Suppose in addition that the both operators are bounded in $L_p(\mathbb{D}_+^{n+1})$ satisfying the estimates

$$\|\widetilde{T}f\|_{p;\mathbb{D}_+^{n+1}} \le C\|f\|_{p;\mathbb{D}_+^{n+1}}, \quad \|\widetilde{T}_a f\|_{p;\mathbb{D}_+^{n+1}} \le C\|a\|_*\|f\|_{p;\mathbb{D}_+^{n+1}} \tag{4.28}$$

with constants independent of a and f. The following assertions can be proved in the same manner as in §3.

Lemma 4.1. *Let $f\in L_p^{\rm loc}(\mathbb{D}_+^{n+1})$, $p\in(1,\infty)$ and for all $(x^0,r)\in\mathbb{R}^{n-1}\times\mathbb{R}_+$*

$$\int_r^\infty s^{-\frac{n+2}{p}-1}\|f\|_{p;\mathcal{E}_s^+(x^0)}ds<\infty. \tag{4.29}$$

If $\widetilde{T}$ is bounded on $L_p(\mathbb{D}_+^{n+1})$ then

$$\|\widetilde{T}f\|_{p;\mathcal{E}_r^+(x^0)} \le C\,r^{\frac{n+2}{p}}\int_{2r}^\infty s^{-\frac{n+2}{p}-1}\|f\|_{p;\mathcal{E}_s^+(x^0)}ds \tag{4.30}$$

where the constant C is independent of r, x^0, and f.

Theorem 4.2. *Let φ be a weight function satisfying* (3.17) *and $\widetilde{T}$ be a sublinear operator satisfying* (4.26) *and* (4.28)*. Then it is bounded in $M_{p,\varphi}(\mathbb{D}_+^{n+1})$, $p \in (1,\infty)$ and*

$$\|\widetilde{T} f\|_{p,\varphi;\mathbb{D}_+^{n+1}} \le C\|f\|_{p,\varphi;\mathbb{D}_+^{n+1}} \tag{4.31}$$

with a constant C independent of f.

Lemma 4.3. *Let $p \in (1,\infty)$, $a \in BMO(\mathbb{D}_+^{n+1})$, and $\widetilde{T}_a$ satisfy* (4.27) *and* (4.28)*. Suppose that for all $f \in L_p^{\rm loc}(\mathbb{D}_+^{n+1})$*

$$\int_r^\infty \Big(1+\ln\frac{s}{r}\Big) s^{-\frac{n+2}{p}-1}\|f\|_{p;\mathcal{E}_s^+(x^0)} ds < \infty \quad \forall\ (x^0,r) \in \mathbb{R}^{n-1}\times\mathbb{R}_+ . \tag{4.32}$$

Then

$$\|\widetilde{T}_a f\|_{p;\mathcal{E}_r^+(x^0)} \le C\|a\|_* r^{\frac{n+2}{p}} \int_{2r}^\infty \Big(1+\ln\frac{s}{r}\Big)\|f\|_{p;\mathcal{E}_s^+(x^0)} \frac{ds}{s^{\frac{n+2}{p}+1}}$$

with a constant C independent of a,f, x^0 and r.

Theorem 4.4. *Let $p \in (1,\infty)$, $a \in BMO(\mathbb{D}_+^{n+1})$, $\varphi(x^0,r)$ be a weight function satisfying* (3.24) *and $\widetilde{T}_a$ be a sublinear operator satisfying* (3.6) *and* (3.7)*. Then $\widetilde{T}_a$ is bounded in $M_{p,\varphi}(\mathbb{D}_+^{n+1})$, and*

$$\|\widetilde{T}_a f\|_{p,\varphi;\mathbb{D}_+^{n+1}} \le C\|a\|_*\, \|f\|_{p,\varphi;\mathbb{D}_+^{n+1}} \tag{4.33}$$

with a constant C independent of a and f.

5. Singular and nonsingular integrals in generalized Morrey spaces

In the present section we apply the above results to Calderón-Zygmund type operators with parabolic kernel. Since these operators are sublinear and bounded in $L_p(\mathbb{R}^{n+1})$ their continuity in $M_{p,\varphi}$ follows immediately.

Definition 5.1. *A measurable function $\mathcal{K}(x,\xi) : \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}$ is called variable parabolic Calderón-Zygmund kernel if:*

- *i)* *$\mathcal{K}(x,\cdot)$ is a parabolic Calderón-Zygmund kernel for a.a. $x \in \mathbb{R}^{n+1}$:*
  - *a)* $\mathcal{K}(x,\cdot) \in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$,
  - *b)* $\mathcal{K}(x,\mu\xi) = \mu^{-n-2}\mathcal{K}(x,\xi) \quad \forall\mu>0$,
  - *c)* $\displaystyle\int_{\mathbb{S}^n}\mathcal{K}(x,\xi)d\sigma_\xi = 0\,, \quad \int_{\mathbb{S}^n}|\mathcal{K}(x,\xi)|d\sigma_\xi < +\infty.$
- *ii)* $\Big\|D_\xi^\beta\mathcal{K}\Big\|_{L_\infty(\mathbb{R}^{n+1}\times\mathbb{S}^n)} \le M(\beta) < \infty$ *for every multi-index β.*

Moreover

$$|\mathcal{K}(x,x-y)| \le \rho(x-y)^{-n-2}\big|\mathcal{K}\big(x,\frac{x-y}{\rho(x-y)}\big)\big| \le \frac{M}{\rho(x-y)^{n+2}}$$

which means that the singular integrals

$$\begin{cases} \mathfrak{K}f(x) = P.V.\displaystyle\int_{\mathbb{R}^{n+1}}\mathcal{K}(x,x-y)f(y)dy \\ \mathfrak{C}[a,f](x) = P.V.\displaystyle\int_{\mathbb{R}^{n+1}}\mathcal{K}(x,x-y)[a(y)-a(x)]f(y)dy \end{cases} \tag{5.34}$$

are sublinear and bounded in $L_p(\mathbb{R}^{n+1})$ according to the results in [3, 7]. Let us note that any weight function φ satisfying (3.24) satisfies also (3.17) and hence the following holds as a simple application of the estimates proved in §3.

Theorem 5.2. *For any $f \in M_{p,\varphi}(\mathbb{R}^{n+1})$ with (p,φ) as in Theorem 3.7 and $a \in BMO$ there exist constants depending on n, p and the kernel such that*

$$\|\mathfrak{K}f\|_{p,\varphi;\mathbb{R}^{n+1}} \le C\|f\|_{p,\varphi;\mathbb{R}^{n+1}}, \quad \|\mathfrak{C}[a,f]\|_{p,\varphi;\mathbb{R}^{n+1}} \le C\|a\|_*\|f\|_{p,\varphi;\mathbb{R}^{n+1}}. \tag{5.35}$$

Corollary 5.3. *Let Q be a cylinder in $\mathbb{R}^{n+1}_+$, $f \in M_{p,\varphi}(Q)$, $a \in BMO(Q)$ and $\mathcal{K}(x,\xi) : Q \times \mathbb{R}^{n+1}_+ \setminus \{0\} \to \mathbb{R}$. Then the operators (5.34) are bounded in $M_{p,\varphi}(Q)$ and*

$$\|\mathfrak{K}f\|_{p,\varphi;Q} \le C\|f\|_{p,\varphi;Q}, \quad \|\mathfrak{C}[a,f]\|_{p,\varphi;Q} \le C\|a\|_*\|f\|_{p,\varphi;Q} \tag{5.36}$$

with C independent of a and f.

Proof. Define the extensions

$$\overline{\mathcal{K}}(x,\xi) = \begin{cases} \mathcal{K}(x,\xi) & (x,\xi) \in Q \times \mathbb{R}^{n+1}_+ \setminus \{0\} \\ 0 & \text{elsewhere} \end{cases}, \quad \overline{f}(x) = \begin{cases} f(x) & x \in Q \\ 0 & x \notin Q. \end{cases}$$

Denote by $\overline{\mathfrak{K}}f$ the singular integral with a kernel $\overline{\mathcal{K}}$ and potential $\overline{f}$. Then

$$|\mathfrak{K}f| \le |\overline{\mathfrak{K}}f| \le C\int_{\mathbb{R}^{n+1}} \frac{|\overline{f}(y)|}{\rho(x-y)^{n+2}}\,dy$$

and

$$\|\mathfrak{K}f\|_{p,\varphi;Q} \le \|\overline{\mathfrak{K}}f\|_{p,\varphi;\mathbb{R}^{n+1}} \le C\|\overline{f}\|_{p,\varphi;\mathbb{R}^{n+1}} = C\|f\|_{p,\varphi;Q}.$$

The estimate for the commutator follows in a similar way. □

Corollary 5.4. *Let $a \in VMO$ and (p,φ) be as in Theorem 3.7. Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon, \eta_a)$ such that for any $\mathcal{E}_r(x_0)$ with a radius $r \in (0, r_0)$ and all $f \in M_{p,\varphi}(\mathcal{E}_r(x_0))$*

$$\|\mathfrak{C}[a,f]\|_{p,\varphi;\mathcal{E}_r(x_0)} \le C\varepsilon\|f\|_{p,\varphi;\mathcal{E}_r(x_0)} \tag{5.37}$$

where C is independent of ε, f, r and x_0.

Proof. Since any VMO function can be approximated by BUC functions (see [6, 21]) for each $\varepsilon > 0$ there exists $r_0(\varepsilon, \eta_a)$ and $g \in BUC$ with modulus of continuity $\omega_g(r_0) < \varepsilon/2$ such that $\|a - g\|_* < \varepsilon/2$. Fixing $\mathcal{E}_r(x_0)$ with $r \in (0, r_0)$ define the function

$$h(x) = \begin{cases} g(x) & x \in \mathcal{E}_r(x_0) \\ g\big(x_0 + r\dfrac{x' - x_0'}{\rho(x-x_0)}, t_0 + r^2\dfrac{t - t_0}{\rho^2(x-x_0)}\big) & x \in \mathcal{E}_r^c(x_0) \end{cases}$$

such that $h \in BUC(\mathbb{R}^{n+1})$ and $\omega_h(r_0) \le \omega_g(r_0) < \varepsilon/2$. Hence

$$\begin{aligned} \|\mathfrak{C}[a,f]\|_{p,\varphi;\mathcal{E}_r(x_0)} &\le \|\mathfrak{C}[a-g,f]\|_{p,\varphi;\mathcal{E}_r(x_0)} + \|\mathfrak{C}[g,f]\|_{p,\varphi;\mathcal{E}_r(x_0)} \\ &\le C\|a-g\|_*\|f\|_{p,\varphi;\mathcal{E}_r(x_0)} + \|\mathfrak{C}[h,f]\|_{p,\varphi;\mathcal{E}_r(x_0)} < C\varepsilon\|f\|_{p,\varphi;\mathcal{E}_r(x_0)}\,. \end{aligned}$$

□

For any $x' \in \mathbb{R}^n_+$ and any fixed $t > 0$ define the *generalized reflection*

$$\mathcal{T}(x) = (\mathcal{T}'(x), t) \qquad \mathcal{T}'(x) = x' - 2x_n\frac{\mathbf{a}^n(x',t)}{a^{nn}(x',t)} \tag{5.38}$$

where $\mathbf{a}^n(x)$ is the last row of the coefficients matrix $\mathbf{a}(x)$ of (2.1). The function $\mathcal{T}'(x)$ maps $\mathbb{R}^n_+$ into $\mathbb{R}^n_-$ and the kernel $\mathcal{K}(x, \mathcal{T}(x) - y) = \mathcal{K}(x, \mathcal{T}'(x) - y', t - \tau)$ is

nonsingular one for any $x, y \in \mathbb{D}_+^{n+1}$. Taking $\widetilde{x} \in \mathbb{D}_-^{n+1}$ there exist positive constants κ_1 and κ_2 such that

$$\kappa_1 \rho(\widetilde{x} - y) \le \rho(\mathcal{T}(x) - y) \le \kappa_2 \rho(\widetilde{x} - y). \tag{5.39}$$

For any $f \in M_{p,\varphi}(\mathbb{D}_+^{n+1})$ and $a \in BMO(\mathbb{D}_+^{n+1})$ define the nonsingular integral operators

$$\begin{cases} \widetilde{\mathfrak{K}} f(x) = \displaystyle\int_{\mathbb{D}_+^{n+1}} \mathcal{K}(x, \mathcal{T}(x) - y) f(y) dy \\ \widetilde{\mathfrak{C}}[a, f](x) = \displaystyle\int_{\mathbb{D}_+^{n+1}} \mathcal{K}(x, \mathcal{T}(x) - y)[a(y) - a(x)] f(y) dy. \end{cases} \tag{5.40}$$

Since $\mathcal{K}(x, \mathcal{T}(x) - y)$ is still homogeneous one and satisfies the conditin $b)$ in Definition 5.1 we have

$$|\mathcal{K}(x, \mathcal{T}(x) - y)| \le \frac{M}{\rho(\mathcal{T}(x) - y)^{n+2}} \le \frac{C}{\rho(\widetilde{x} - y)^{n+2}}.$$

Hence the operators (5.40) are sublinear and bounded in $L_p(\mathbb{D}_+^{n+1})$, $p \in (1, \infty)$ (cf. [3]). The following estimates are simple consequence of the results in §4.

Theorem 5.5. *Let $a \in BMO(\mathbb{D}_+^{n+1})$ and $f \in M_{p,\varphi}(\mathbb{D}_+^{n+1})$ with (p, φ) as in Theorem 3.7. Then the operators $\widetilde{\mathfrak{K}} f$ and $\widetilde{\mathfrak{C}}[a, f]$ are continuous in $M_{p,\varphi}(\mathbb{D}_+^{n+1})$ and*

$$\|\widetilde{\mathfrak{K}} f\|_{p,\varphi;\mathbb{D}_+^{n+1}} \le C\|f\|_{p,\varphi;\mathbb{D}_+^{n+1}}, \quad \|\widetilde{\mathfrak{C}}[a, f]\|_{p,\varphi;\mathbb{D}_+^{n+1}} \le C\|a\|_* \|f\|_{p,\varphi;\mathbb{D}_+^{n+1}} \tag{5.41}$$

with a constant independend of a and f.

Corollary 5.6. *Let $a \in VMO$ and (p, φ) be as above. Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon, \eta_a)$ such that for any $\mathcal{E}_r^+(x^0)$ with a radius $r \in (0, r_0)$ and all $f \in M_{p,\varphi}(\mathcal{E}_r^+(x^0))$*

$$\|\mathfrak{C}[a, f]\|_{p,\varphi;\mathcal{E}_r^+(x^0)} \le C\varepsilon \|f\|_{p,\varphi;\mathcal{E}_r^+(x^0)}, \tag{5.42}$$

where C is independent of ε, f, r and x^0.

6. Proof of the main result

Consider the problem (2.1) with $f \in M_{p,\varphi}(Q)$, (p, φ) as in Theorem 3.7. Since $M_{p,\varphi}(Q)$ is a proper subset of $L_p(Q)$ than (2.1) is uniquely solvable and the solution u belongs at least to $\overset{\circ}{W}{}_p^{2,1}(Q)$. Our aim is to show that this solution belongs also to $\overset{\circ}{W}{}_{p,\varphi}^{2,1}(Q)$. For this goal we need a priori estimate of u that we are going to prove in two steps.

Interior estimate. For any $x_0 \in \mathbb{R}_+^{n+1}$ consider the parabolic semi-cylinders $\mathcal{C}_r(x_0) = \mathcal{B}_r(x_0') \times (t_0 - r^2, t_0)$. Let $v \in C_0^\infty(\mathcal{C}_r)$ and suppose that $v(x, t) = 0$ for $t \le 0$. According to [3, Theorem 1.4] for any $x \in \operatorname{supp} v$ the following representation formula for the second derivatives of v holds true

$$\begin{aligned} D_{ij} v(x) =& P.V. \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x, x - y)[a^{hk}(y) - a^{hk}(x)] D_{hk} v(y) dy \\ &+ P.V. \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x, x - y) \mathcal{P} v(y) dy + \mathcal{P} v(x) \int_{\mathbb{S}^n} \Gamma_j(x, y) \nu_i d\sigma_y, \end{aligned} \tag{6.43}$$

where $\nu(\nu_1, \dots, \nu_{n+1})$ is the outward normal to $\mathbb{S}^n$. Here $\Gamma(x, \xi)$ is the fundamental solution of the operator $\mathcal{P}$ and $\Gamma_{ij}(x, \xi) = \partial^2 \Gamma(x, \xi)/\partial \xi_i \partial \xi_j$. Since any function

$v \in W_p^{2,1}$ can be approximated by C_0^∞ functions, the representation formula (6.43) still holds for any $v \in W_p^{2,1}(\mathcal{C}_r(x_0))$. The properties of the fundamental solution (cf. [3, 15, 23]) imply Γ_{ij} are variable Calderón-Zygmund kernels in the sense of Definition 5.1. Using the notations (5.34) we can write

$$\begin{aligned} D_{ij}v(x) = & \mathfrak{C}_{ij}[a^{hk}, D_{hk}v](x) \\ & + \mathfrak{K}_{ij}(\mathcal{P}v)(x) + \mathcal{P}v(x)\int_{\mathbb{S}^n} \Gamma_j(x,y)\nu_i d\sigma_y\,. \end{aligned} \tag{6.44}$$

The integrals $\mathfrak{K}_{ij}$ and $\mathfrak{C}_{ij}$ are defined by (5.34) with kernels $\mathcal{K}(x, x-y) = \Gamma_{ij}(x, x-y)$. Because of Corollaries 5.3 and 5.4 and the equivalence of the metrics we get

$$\|D^2 v\|_{p,\varphi;\mathcal{C}_r(x_0)} \le C(\varepsilon\|D^2 v\|_{p,\varphi;\mathcal{C}_r(x_0)} + \|\mathcal{P}u\|_{p,\varphi;\mathcal{C}_r(x_0)}) \tag{6.45}$$

for some r small enough. Moving the norm of D^2v on the left-hand side we get

$$\|D^2 v\|_{p,\varphi;\mathcal{C}_r(x_0)} \le C(n,p,\eta_a(r),\|D\Gamma\|_{\infty,Q})\|\mathcal{P}v\|_{p,\varphi;\mathcal{C}_r(x_0)}.$$

Define a cut-off function $\phi(x) = \phi_1(x')\phi_2(t)$, with $\phi_1 \in C_0^\infty(\mathcal{B}_r(x'_0))$, $\phi_2 \in C_0^\infty(\mathbb{R})$ such that

$$\phi_1(x') = \begin{cases} 1 & x' \in \mathcal{B}_{\theta r}(x'_0) \\ 0 & x' \notin \mathcal{B}_{\theta' r}(x'_0) \end{cases}, \qquad \phi_2(t) = \begin{cases} 1 & t \in (t_0 - (\theta r)^2, t_0] \\ 0 & t < t_0 - (\theta' r)^2 \end{cases}$$

with $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > \theta$ and $|D^s\phi| \le C[\theta(1-\theta)r]^{-s}$, $s = 0,1,2$, $|\phi_t| \sim |D^2\phi|$. For any solution $u \in W_p^{2,1}(Q)$ of (2.1) define $v(x) = \phi(x)u(x) \in W_p^{2,1}(\mathcal{C}_r)$. Hence

$$\begin{aligned} \|D^2 u\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)} &\le \|D^2 v\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} \le C\|\mathcal{P}v\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} \\ &\le C\left(\|f\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} + \frac{\|Du\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)}}{\theta(1-\theta)r} + \frac{\|u\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)}}{[\theta(1-\theta)r]^2}\right). \end{aligned}$$

Hence

$$\begin{aligned} &\big[\theta(1-\theta)r\big]^2\|D^2 u\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)} \\ &\qquad \le \left([\theta(1-\theta)r]^2\|f\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} + \theta(1-\theta)r\|Du\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} + \|u\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)}\right) \\ &\qquad\quad \text{(by the choice of } \theta' \text{ it follows } \theta(1-\theta) \le 2\theta'(1-\theta')) \\ &\qquad \le C\left(r^2\|f\|_{p,\varphi;Q} + \theta'(1-\theta')r\|Du\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)} + \|u\|_{p,\varphi;\mathcal{C}_{\theta' r}(x_0)}\right). \end{aligned}$$

Introducing the semi-norms

$$\Theta_s = \sup_{0<\theta<1}\big[\theta(1-\theta)r\big]^s\|D^s u\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)} \qquad s = 0,1,2$$

the above inequality becomes

$$[\theta(1-\theta)r]^2\|D^2 u\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)} \le \Theta_2 \le C\left(r^2\|f\|_{p,\varphi;Q} + \Theta_1 + \Theta_0\right). \tag{6.46}$$

The interpolation inequality [24, Lemma 4.2] gives that there exists a positive constant C independent of r such that

$$\Theta_1 \le \varepsilon\,\Theta_2 + \frac{C}{\varepsilon}\,\Theta_0 \qquad \text{for any } \varepsilon \in (0,2).$$

Thus (6.46) becomes

$$[\theta(1-\theta)r]^2\|D^2 u\|_{p,\varphi;\mathcal{C}_{\theta r}(x_0)} \le \Theta_2 \le C\left(r^2\|f\|_{p,\varphi;Q} + \Theta_0\right) \quad \forall\ \theta \in (0,1).$$

Taking $\theta = 1/2$ we get the Caccioppoli-type estimate

$$\|D^2u\|_{p,\varphi;\mathcal{C}_{r/2}(x_0)} \le C\left(\|f\|_{p,\varphi;Q} + \frac{1}{r^2}\|u\|_{p,\varphi;\mathcal{C}_r(x_0)}\right).$$

To estimate u_t we exploit the parabolic structure of the equation and the boundedness of the coefficients

$$\begin{aligned}\|u_t\|_{p,\varphi;\mathcal{C}_{r/2}(x_0)} &\le \|\mathbf{a}\|_{\infty;Q}\|D^2u\|_{p,\varphi;\mathcal{C}_{r/2}(x_0)} + \|f\|_{p,\varphi;\mathcal{C}_{r/2}(x_0)}\\ &\le C\big(\|f\|_{p,\varphi;Q} + \frac{1}{r^2}\|u\|_{p,\varphi;\mathcal{C}_r(x_0)}\big).\end{aligned}$$

Consider cylinders $Q' = \Omega' \times (0,T)$ and $Q'' = \Omega'' \times (0,T)$ with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, by standard covering procedure and partition of the unity we get

$$\|u\|_{W^{2,1}_{p,\varphi}(Q')} \le C\big(\|f\|_{p,\varphi;Q} + \|u\|_{p,\varphi;Q''}\big) \tag{6.47}$$

where C depends on $n, p, \Lambda, T, \|D\Gamma\|_{\infty;Q}, \eta_{\mathbf{a}}(r)$, $\|\mathbf{a}\|_{\infty,Q}$ and $\operatorname{dist}(\Omega', \partial\Omega'')$.

Boundary estimates. For any fixed $(x^0, r) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ define the semi-cylinders

$$\mathcal{C}^+_r(x^0) = \mathcal{B}^+_r(x^{0\prime}) \times (0, r^2) = \{|x^0 - x'| < r, x_n > 0, 0 < t < r^2\}$$

with $SS^+_r = \{(x'', 0, t) : |x^0 - x''| < r, 0 < t < r^2\}$. For any solution $u \in W^{2,1}_p(\mathcal{C}^+_r(x^0))$ with $\operatorname{supp} u \in \mathcal{C}^+_r(x^0)$ the following boundary representation formula holds (cf. [3])

$$\begin{aligned}D_{ij}u(x) =& \mathfrak{C}_{ij}[a^{hk}, D_{hk}u](x) + \mathfrak{K}_{ij}(\mathcal{P}u)(x)\\ &+ \mathcal{P}u(x)\int_{\mathbb{S}^n}\Gamma_j(x,y)\nu_i d\sigma_y - \mathfrak{I}_{ij}(x)\end{aligned}$$

where

$$\mathfrak{I}_{ij}(x) = \widetilde{\mathfrak{K}}_{ij}(\mathcal{P}u)(x) + \widetilde{\mathfrak{C}}_{ij}[a^{hk}, D_{hk}u](x), \quad i,j = 1,\ldots,n-1,$$

$$\mathfrak{I}_{in}(x) = \mathfrak{I}_{ni}(x) = \sum_{l=1}^{n}\left(\frac{\partial\mathcal{T}(x)}{\partial x_n}\right)^l\left[\widetilde{\mathfrak{C}}_{il}[a^{hk}, D_{hk}u](x) + \widetilde{\mathfrak{K}}_{il}(\mathcal{P}u)(x)\right], \quad i = 1,\ldots,n-1,$$

$$\mathfrak{I}_{nn}(x) = \sum_{r,l=1}^{n}\left(\frac{\partial\mathcal{T}(x)}{\partial x_n}\right)^r\left(\frac{\partial\mathcal{T}(x)}{\partial x_n}\right)^l\left[\widetilde{\mathfrak{C}}_{rl}[a^{hk}, D_{hk}u](x) + \widetilde{\mathfrak{K}}_{rl}(\mathcal{P}u)(x)\right],$$

$$\frac{\partial\mathcal{T}(x)}{\partial x_n} = \left(-2\frac{a^{n1}(x)}{a^{nn}(x)}, \ldots, -2\frac{a^{nn-1}(x)}{a^{nn}(x)}, -1, 0\right).$$

Here $\widetilde{\mathfrak{K}}_{ij}$ and $\widetilde{\mathfrak{C}}_{ij}$ are the operators defined by (5.40) with kernels $\mathcal{K}(x, \mathcal{T}(x) - y) = \Gamma_{ij}(x, \mathcal{T}(x) - y)$. Applying the estimates (5.41) and (5.42) and having in mind that the components of the vector $\frac{\partial\mathcal{T}(x)}{\partial x_n}$ are bounded we get

$$\|D^2u\|_{p,\varphi;\mathcal{C}^+_r(x^0)} \le C\big(\|\mathcal{P}u\|_{p,\varphi;\mathcal{C}^+_r(x^0)} + \|u\|_{p,\varphi;\mathcal{C}^+_r(x^0)}\big).$$

The Jensen inequality applied to $u(x) = \int_0^t u_s(x', s)ds$ and the parabolic structure of the equation give

$$\|u\|_{p,\varphi;\mathcal{C}^+_r(x^0)} \le Cr^2\|u_t\|_{p,\varphi;\mathcal{C}^+_r(x^0)} \le C(\|f\|_{p,\varphi;Q} + r^2\|u\|_{p,\varphi;\mathcal{C}^+_r(x^0)}).$$

Taking r small enough we can move the norm of u on the left-hand side obtaining

$$\|u\|_{p,\varphi;\mathcal{C}^+_r} \le C\|f\|_{p,\varphi;Q}$$

with a constant C depending on $n, p, \Lambda, T, \eta_{\mathbf{a}}, \|\mathbf{a}\|_{\infty,Q}$. By covering of the boundary with small cylinders, partition of the unit subordinated of that covering and local flattering of $\partial\Omega$ we get that

$$\|u\|_{W^{2,1}_{p,\varphi}(Q\setminus Q')} \le C\|f\|_{p,\varphi;Q}\,. \tag{6.48}$$

Unifying (6.47) and (6.48) we get (2.4).

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[1]Ahi Evran University, Department of Mathematics, Kirsehir, Turkey, Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku

E-mail address: vagif@guliyev.com

[2]Department of Civil Engineering, Second University of Naples, Via Roma 29, 81031 Aversa, Italy

E-mail address: luba.softova@unina2.it